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Products of three triangular matrices over commutative rings

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Abstract

We characterize commutative rings with 1 over which any $n \times n$ matrix A can be written as LUM, where L, M are lower triangular, U is upper triangular and L and U have all their diagonal entries 1. © 2002 Elsevier Science Inc. All rights reserved.

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0. Introduction

Vaserstein and Wheland [4] proved the interesting theorem that if R is any ring (not necessarily commutative) with Bass stable rank 1, then every invertible $n \times n$ matrix A over R can be written as $A = \lambda\rho\mu$, where λ, μ are lower triangular and ρ is upper triangular.

We show that for commutative rings the converse is true: If every invertible 2×2 matrix over a commutative ring R is a product $\lambda\rho\mu$ as above, then R has Bass stable rank 1.

Throughout this paper, R stands for a commutative ring with 1 and $M_n(R)$ for the set of all $n \times n$ matrices with coefficients in R.

In [3] it was proved that *any* $n \times n$ matrix A over a field F can be written as $A = LUM$, where L, M are lower triangular matrices, U is an upper triangular matrix and in L and U all the diagonal entries are equal to 1.

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We prove that, every $A \in M_n(R)$ can be written as $A = LUM$ as above if and only if R is of Bass stable rank 1 and R is a Hermite ring (in the sense of Kaplansky [2]). Valuation rings, von Neumann regular rings, the ring of all entire functions, the ring of all algebraic integers and also the arbitrary product of such rings satisfy the above two conditions [2,5].

In [2, Theorem 3.5], Kaplansky has shown that if R is a Hermite ring and $A \in M_n(R)$, then there is an invertible matrix Q such that AQ is a lower triangular (or an upper triangular) matrix.

For our purpose we require $A = PL$, P invertible and L lower triangular. We show that these forms are equivalent and equivalent to R being a Hermite ring (see Theorem 2.5). This, perhaps is also implicit in [2]. We point out another easy characterization of a Hermite ring which is implicit in [2]. (See Proposition 2.2.)

1. Converse of Vaserstein and Wheland's Theorem

Definition 1.1. R has Bass stable rank 1 if for any two elements a, b of R the condition $Ra + Rb = R$ implies that there exists an $x \in R$ such that $a + xb$ is a unit in R.

The set of all lower triangular matrices in $M_n(R)$ is denoted by $\mathscr{L}$. The set of all upper triangular matrices in $M_n(R)$ is denoted by $\mathscr{U}$. We write $\mathscr{L}\mathscr{U}\mathscr{L} = \{\lambda\rho\mu\colon \lambda, \mu \in \mathscr{L},\ \rho \in \mathscr{U}\}$.

Lemma 1.2. *Suppose*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ v & w \end{pmatrix}$$

where $ad - bc$ is a unit in R and $b \neq 0$, $c \neq 0$. Then $d - xb$ is a unit in R.

Proof. Equating entries we get

$$a = u + yv, \quad b = yw,$$
$$c = x(u + yv) + v = xa + v, \quad d = xyw + w = xb + w.$$

Hence $w = d - xb$.

Now

$$\begin{aligned} ad - bc &= a(xb + w) - b(xa + v) \\ &= aw - bv \\ &= aw - yvw \\ &= (a - yv)w \\ &= uw. \end{aligned}$$

Hence, $w = d - xb$ is a unit and the lemma follows. □

Proposition 1.3. *If every invertible element of $M_2(R)$ belongs to $\mathscr{L}\mathscr{U}\mathscr{L}$, then R has Bass stable rank* 1.

Proof. Let $d, b \in R$ with $Rd + Rb = R$. We prove that $d - xb$ is a unit in R for some $x \in R$.

We have $1 = ad - bc$ for some $a, c \in R$.

Now

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = LUL_1, \quad L, L_1 \in \mathscr{L}, \ U \in \mathscr{U}.$$

Each L, U, L_1 is invertible and hence diagonal elements of L, U, L_1 are all units in R. By interpolating diagonal matrices (see [3,Theorem 1.3]) without loss of generality we can assume

$$L = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}.$$

Hence by Lemma 1.2, $d - xb$ is a unit of R. □

The following lemma provides an explicit factorization.

Lemma 1.4. *Let*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $d - xb$ a unit in R. Then

$$A = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} 1 & b(d - xb)^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (ad - bc)(d - xb)^{-1} & 0 \\ c - xa & d - xb \end{pmatrix}$$

Proof. The proof is an easy verification. □

2. Hermite rings

Definition 2.1 [2]. R is said to be a *Hermite ring* if given a 1×2 matrix $(a \ \ b)$ there exists an invertible $Q \in M_2(R)$ such that $(a \ \ b)Q = (d \ \ 0)$.

The following easy proposition is implicit in [2].

Proposition 2.2. *The following two conditions are equivalent*:

(i) *R is a Hermite ring*;

(ii) *whenever $a, b \in R$ there exist elements a_1, b_1, c_1, d_1 and d in R such that $a = a_1 d$, $b = b_1 d$ and $a_1 d_1 - b_1 c_1$ is a unit in R.*

Proof. (i) $\Rightarrow$ (ii): Let $a, b \in R$. Let $Q \in M_2(R)$ be invertible such that $(a \ \ b)Q = (d \ \ 0)$. Then $(a \ \ b) = (d \ \ 0)Q^{-1}$. Let

$$Q^{-1} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}.$$

Then $a = a_1 d,\ b = b_1 d$, and $a_1 d_1 - b_1 c_1$ is a unit.

(ii) $\Rightarrow$ (i): Let $a, b \in R$. Then

$$(a \quad b) = (d \quad 0) \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

and hence if Q is the inverse of

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix},$$

then $(a \ \ b)Q = (d \ \ 0)$. Thus we get R is a Hermite ring. $\square$

Definition 2.3 [2]. We say Hermite triangular form can be achieved in R if given $A \in M_n(R)$ there exists an invertible $Q \in M_n(R)$ such that AQ is a lower triangular matrix.

Kaplansky proved that if R is a Hermite ring, then Hermite triangular form can be achieved in R [2, Theorem 3.5]. We need the following.

Definition 2.4. We say that the *PL* theorem holds in R if given $A \in M_n(R)$ there exists an invertible $P \in M_n(R)$ and a lower triangular matrix L such that $A = PL$.

The following theorem is perhaps implicit in [2].

Theorem 2.5. *The following conditions are equivalent*:

(a) *R is a Hermite ring.*

(b) *Hermite triangular form can be achieved in R.*

(c) *The PL theorem holds in R.*

Proof. (a) $\Rightarrow$ (b) is Theorem 3.5 of [2].

(b) $\Rightarrow$ (c): Consider $M_n(R)$.

Let

$$T = \begin{pmatrix} 0 & 0 & \cdot & \cdot & \cdot & 1 \\ 0 & 0 & \cdot & \cdot & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}.$$

Then, $T^2 = \mathrm{Id}$, and if $A = (a_{ij})$ and $TAT = (b_{ij})$, it is easily verified that $b_{ij} = a_{(n+1-i)(n+1-j)}$. It follows that if $A \in \mathscr{L}$, then $TAT \in \mathscr{U}$ and if $A \in \mathscr{U}$, then $TAT \in \mathscr{L}$.

Let $A \in M_n(R)$ be any matrix. Put $B = TA^{\mathrm{t}}T$, where A^{t} is the transpose of A. By (b) there exists an invertible matrix Q and a lower triangular matrix L such that $BQ = L$. Let $S = Q^{-1}$ so that $B = LS$. Then $TBT = A^{\mathrm{t}} = (TLT)(TST)$. Hence $A = (TST)^{\mathrm{t}}(TLT)^{\mathrm{t}} = PL_1$, where P is invertible and L_1 is a lower triangular matrix.

(c) $\Rightarrow$ (a): Let $a, b \in R$. by (c) we have

$$\begin{pmatrix} 0 & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ \beta & \delta \end{pmatrix},$$

where $xw - yz = u$ is a unit in R. This gives $a = w\delta$ and $b = y\delta$. Now

$$\begin{aligned} (a \quad b) \begin{pmatrix} x & -y \\ -z & w \end{pmatrix} &= \begin{pmatrix} ax - bz & bw - ay \end{pmatrix} \\ &= \begin{pmatrix} u\delta & 0 \end{pmatrix} \\ &= \begin{pmatrix} d & 0 \end{pmatrix}, \end{aligned}$$

where $d = u\delta$. Thus R is a Hermite ring. □

3. The main theorem

Theorem 3.1. *The following conditions are equivalent*:

(1) *If $A \in M_n(R)$, then A can be written as $A = LUM$, $L \in \mathscr{L}$, $U \in \mathscr{U}$, $M \in \mathscr{L}$ and in L and U all the diagonal entries are equal to* 1.

(2) *R has Bass stable rank* 1 *and R is a Hermite ring.*

Proof. (1) $\Rightarrow$ (2): By Proposition 1.3 R has Bass stable rank 1. If $A = LUM$, then (LU) is invertible and $M \in \mathscr{L}$. Hence PL theorem holds in R. By Theorem 2.5, R is a Hermite ring.

(2) $\Rightarrow$ (1): Let $A \in M_n(R)$. Then by Theorem 2.5, $A = PL$, P is invertible in $M_n(R)$ and $L \in \mathscr{L}$.

By the theorem of Vaserstein and Wheland [4, Theorem 1], $P = BCD$, $B \in \mathscr{L}$, $C \in \mathscr{U}$, $D \in \mathscr{L}$. Since P is invertible, $\det P$ is a unit and so $\det B$, $\det C$, $\det D$ are units, and hence B, C, D are all invertible. This shows that the diagonal entries in B, C, D are all units. Hence by successively interpolating diagonal matrices (first between B and C) we can assume that diagonal entries in B and C are all 1 (see [3, Theorem 1.3]).

Now $A = BCDL = BC(DL)$ which is the product form required in (1). Thus (1) holds. This proves the theorem. □

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